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A GENERAL NONLINEAR THEORY OF ELASTIC SHELLS

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Abstract—This paper presents a general nonlinear theory of elastic shells for large deflections and finite strains in reference to a certain natural state. By expanding the displacement components into power series in the coordinate θ^3 normal to the undeformed middle surface of shells, the expansions of the Cauchy–Green strain tensors are expressed in terms of these expanded displacement components. Through the modified Hellinger–Reissner variational principle for a three-dimensional elastic continuum, a set of the fundamental shell equations is derived in terms of the expanded Cauchy–Green strain tensors and Kirchhoff stress resultants. The Love–Kirchhoff hypothesis is not assumed and higher order stretching and bending are taken into consideration. For elastic shells of isotropic materials, assuming the strain-energy to be an analytic function of the strain measures, general nonlinear constitutive equations are then derived. Thus, a complete and consistent two-dimensional shell theory incorporating the geometrical and physical nonlinearities is established. The classical theories of shells are directly derivable from the present results by proper truncations of the series.

1. INTRODUCTION

Although a number of significant contributions have been made in the literature, a complete nonlinear theory of shells, which enables one to describe the large elastic deformations of shells undergoing finite strains, is not presently available. A fully consistent two-dimensional nonlinear theory of shells incorporating the geometrical and physical nonlinearities is derived here from the three-dimensional theory of elasticity.

In the two-dimensional theory of shells, a great number of investigations have been developed on the basis of the well-known Love–Kirchhoff hypothesis. Under the assumption of small strains and small displacements, Naghdi[1, 2] has developed a systematic derivation of the fundamental equations in the linear shell theory. There still remains, however, some theoretical problems to be solved, especially so far as shells undergoing large deflections and/or finite strains are concerned. For thin elastic shells undergoing large deflections but small strains, several theories for the geometrically nonlinear problems have been developed. By introducing the Euler stress resultant tensors and the bending strain tensor defined by the difference of the second fundamental tensors of the middle surface in the deformed and undeformed states, Sanders [3] has developed a nonlinear thin shell theory and shown that the existing theories can be derived under various approximations. The equations of equilibrium have been written in the directions of the base vectors of the undeformed middle surface, but an approximation has been made for the relation between the transverse shear and bending moment resultants. Koiter[4] has also derived similar results under the assumptions of small strains and plane state of stresses. However, in these investigations, since the definitions of stress resultant tensors and also of bending strain tensor have been left uncertain from the viewpoint of the three-dimensional theory of elasticity due to the assumption of small strains and the constitutive equations have not been examined, it seems that there exists a certain inconsistency in the order of approximations among the strain–displacement relations, the equations of equilibrium, the boundary conditions and the simplest linear constitutive equations. A consistent approximation of the fundamental shell equations, therefore, seems to be established only if an appropriate set of the constitutive equations are incorporated simultaneously. In [3, 4], the fundamental equations have been expressed with respect to the undeformed state, but the Euler stress resultants and the Cauchy–Green strain tensors which do not correspond exactly from the viewpoint of elastic energy have been made use of due to the assumption of small strains. On the other hand, with the use of the Lagrange stress tensors, Glockner[5] has derived nonlinear equations of equilibrium in reference to the known undeformed state under the Love–Kirchhoff hypothesis. Practical applications of these equations may be limited, because the Lagrange stress tensors are not symmetric. Under these circumstances it is considered indispensable to make use of clearly defined stress and strain tensors and to derive a consistent set of fundamental equations from the viewpoint of elastic energy particularly in the general shell theory for large deflections and finite strains.

An actual shell body is always three-dimensional and therefore any two-dimensional theory of shells is necessarily of an approximate character. In order to estimate the order of approximations and to define strains and stress resultants clearly which are introduced in any two-dimensional theory of shells, it seems necessary to derive a shell theory from the three-dimensional theory of elasticity. In the linear theory, Koiter[6] has presented a definition of the stress resultant tensors in his ealier work[4] in terms of the stress tensors and developed a modified Love–Kirchhoff theory by taking into account the effect of normal strain γ_{33}. An error estimate of this theory as an approximation to the actual three-dimensional solution has been derived.

In view of the two-dimensional character of a shell body, it has been natural to consider an expansion of the three-dimensional equations of the elasticity theory with respect to some small parameter related, for instance, to the thickness curvature ratio. Within the scope of the linear shell theory, Rutten[7] has derived two-dimensional interior shell equations by expanding the components of displacements, stresses and strains into uniformly convergent Taylor series in the coordinate variable θ^3 normal to the middle surface, and derived also the edge-zone equations to analyse the "excess" state in the edge-zone of a shell. For geometrically nonlinear problems, a set of nonlinear shell equations which incorporates the linear constitutive equations for anisotropic materials has been derived by Habip[8], who has clearly introduced the Kirchhoff stress tensors and the Cauchy–Green strain tensors in the reference state. A theory of thick elastic shells obeying the linear constitutive equations has been developed by Martinez–Marquez[9] using power series expansions. In his analysis the Eulerian expressions have been adopted. For thin elastic shells of isotropic homogeneous materials, a set of the fundamental equations in terms of the Cauchy–Green strain tensors and the Kirchhoff stress tensors has been proposed by Sumino[10] under the Love–Kirchhoff hypothesis. An expression of the displacement vector in the nonlinear shell theory under this hypothesis has been assumed and, accordingly, the quadratic terms of the displacement components have been taken into account while their cubic and higher order terms have been neglected. Using thermodynamic considerations, the fundamental equations of thick shells of arbitrary materials and for large deflections have been derived by Krätzig[11] through the variational principle in the Eulerian description. It is apparent from these investigations that the most important merit of the three-dimensional approach is that the Love–Kirchhoff hypothesis will no longer be necessary and that an "exact" shell theory can be derived.

In the most general form of the shell theory for large deflections and finite strains, a nonlinear stress–strain relation must necessarily be introduced into the set of fundamental equations. A number of investigations for general constitutive equations of the three-dimensional elastic body have been published[12–15]. In shell theories, corresponding to the assumption of small strains and small curvature, the simplest linear constitutive equations have always been introduced into geometrically nonlinear problems[3, 4]. Within the assumption of small displacements, nonlinear constitutive equations have been derived by Wainwright[16] under the Love–Kirchhoff hypothesis and also by Librescu[17] without this hypothesis. In these two papers, applicabilities of the constitutive equations will be limited to incompressible materials in which the strain energy function Σ is expressed as a function of two strain invariants I_1 and I_2. For shells of incompressible materials undergoing large elastic deformations, a theory which admits a prescribed thickness change has been given by Biricikoglu and Kalnins[18]. This derivation has been made in terms of the Cauchy–Green strain tensors with the Euler stress resultants and the equations of equilibrium have been written in reference to the directions of the tangents and normal to the deformed middle surface.

Although the Love–Kirchhoff hypothesis is an effective and powerful assumption to express the deformed states of plates and shells, quantitative examinations of this hypothesis have not been published to date. It is true that an elimination of this hypothesis results in the complexity of the governing equations. Within the assumption of small displacements, Librescu[17] has derived a physically nonlinear theory without the Love–Kirchhoff hypothesis. This result indicates that it will be an effective technique to expand the displacement components into power series in θ^3 even in the geometrically nonlinear problems under consideration.

In the present paper, a set of fundamental equations is clearly derived in reference to a known undeformed state (which is considered to be convenient from the viewpoint of practical applications) in terms of the Cauchy–Green strain tensors and the Kirchhoff stress tensors. Through the modified Hellinger–Reissner variational principle of the 3-dimensional elastic body in reference to a natural state, a set of fundamental equations in geometrically nonlinear problems is derived without the Love–Kirchhoff hypothesis by expanding the displacement components into power series in θ^3 and by including terms representing the higher order stretching and bending. In this respect, the present result may be regarded as the Lagrangian formulation corresponding to those in [9, 11], and a generalization of the result in [8] which is based on the assumption that the displacement components are linearly varied through the shell thickness and in [10] which is based on the Love–Kirchhoff hypothesis. The general nonlinear constitutive equations for isotropic elastic materials are also expanded into power series with respect to the strain tensors and the mixed second fundamental tensor of the undeformed middle surface. This result may be regarded as a generalization of that obtained by Librescu[17] in which incompressible materials only are considered. A difference to choose the Euler or Kirchhoff stress tensor will have an influence upon this respect, but the difference diminishes in the linear constitutive equations.

By introducing various approximations with respect to geometrical configurations and deformations of shells and/or material responses, some known approximate theories can be derived from the present results. Throughout the paper, the usual summation convention is used. Repeated Latin indices represent summation over the range (1, 2, 3) and repeated Greek indices, over the range (1, 2).

2. PRELIMINARIES FOR SHELLS IN A NATURAL STATE

A shell is defined as a three-dimensional elastic body of volume V, bounded by the upper (S^+) and lower (S^-) surfaces which are equidistant from the middle surface S and by a lateral surface S which is generated by the normal to S along the bounding curve c. The distance t between S^+ and S^- is called the thickness of the shell. It is assumed that the two external bounding surfaces and the middle surface are continuous and sufficiently smooth, without singularities.

The position of a point on the middle surface of a shell in a natural state may be specified by a set of curvilinear normal coordinates θ^i ($i = 1, 2, 3$), with $\theta^3 = 0$ on S. The spacial position vector may be written in the form:

$$\hat{R}(\theta^1, \theta^2, \theta^3) = \hat{r}(\theta^1, \theta^2) + \theta^3 \hat{a}_3(\theta^1, \theta^2), \tag{2.1}$$

where $\hat{r}(\theta^1, \theta^2)$ denotes the position vector of an arbitrary point on the middle surface and $\hat{a}_3(\theta^1, \theta^2)$, the unit vector normal to the middle surface. The base vector on the middle surface is given by

$$\hat{a}_\alpha = \frac{\partial \hat{r}}{\partial \theta^\alpha} \equiv \hat{r}_{,\alpha}, \quad (\alpha = 1, 2) \tag{2.2}$$

and the metric tensors by

$$a_{\alpha\beta} = \hat{a}_\alpha \cdot \hat{a}_\beta, \qquad a^{\alpha\lambda} a_{\beta\lambda} = \delta^\alpha_\beta, \qquad \hat{a}^\alpha = a^{\alpha\lambda} \hat{a}_\lambda, \tag{2.3}$$

where δ^α_β is the Kronecker symbol. The spacial base vectors and components of the metric tensors are

$$\begin{aligned} &\hat{g}_\alpha = \mu^\lambda_\alpha \hat{a}_\lambda, \qquad \hat{g}^\alpha = (\mu^{-1})^\alpha_\lambda \hat{a}^\lambda, \\ &g_{\alpha\beta} = \hat{g}_\alpha \cdot \hat{g}_\beta = \mu^\lambda_\alpha \mu^\nu_\beta a_{\lambda\nu}, \qquad g^{\alpha\beta} = \hat{g}^\alpha \cdot \hat{g}^\beta = (\mu^{-1})^\alpha_\lambda (\mu^{-1})^\beta_\nu a^{\lambda\nu}, \\ &\hat{g}_3 = \hat{g}^3 = \hat{a}_3 = \hat{a}^3, \qquad g_{33} = g^{33} = a_{33} = a^{33} = 1, \\ &g_{\alpha 3} = g^{\alpha 3} = a_{\alpha 3} = a^{\alpha 3} = 0, \end{aligned} \tag{2.4}$$

where μ^α_β denotes a shifter tensor[1] and is expressed as

$$\mu^\alpha_\beta = \delta^\alpha_\beta - \theta^3 b^\alpha_\beta, \qquad \mu^\alpha_\lambda (\mu^{-1})^\lambda_\beta = \delta^\alpha_\beta. \tag{2.5}$$

The mixed components of the second fundamental tensor of the middle surface, b^α_β is defined by

$$b^\alpha_\beta = b_{\beta\lambda} a^{\alpha\lambda}, \qquad b_{\alpha\beta} = -\hat{a}_{3,\alpha} \cdot \hat{a}_\beta = \hat{a}_3 \cdot \hat{a}_{\alpha,\beta}. \tag{2.6}$$

The element of volume, in terms of normal curvilinear coordinates defined for the middle surface S, is given by

$$\begin{aligned} &\mathrm{d}V = \mu \, \mathrm{d}\theta^3 \, \mathrm{d}S, \qquad \mathrm{d}S = \sqrt{a} \, \mathrm{d}\theta^1 \, \mathrm{d}\theta^2, \\ &\mu = \det(\mu^\alpha_\beta) = \sqrt{\frac{g}{a}} = 1 - 2\theta^3 H + (\theta^3)^2 K, \end{aligned} \tag{2.7}$$

where

$$H = \tfrac{1}{2} b^\alpha_\alpha, \qquad K = \tfrac{1}{2}\delta^{\alpha\beta}_{\lambda\nu} b^\lambda_\alpha b^\nu_\beta, \qquad g = \det(g_{ij}), \qquad a = \det(a_{\alpha\beta}). \tag{2.8}$$

H and K denote, respectively, the mean and Gaussian curvatures and dS is the element of area on the middle surface. Along the edge boundary, the element of area dS and the corresponding unit normal n_α are related as[1]

$$n_\alpha \, \mathrm{dS} = \nu_\alpha \mu \, \mathrm{d}\theta^3 \, \mathrm{d}c, \tag{2.9}$$

where ν_α are components of the outward unit vector normal to the bounding curve c, dc being the line element along this curve.

3. STRAIN–DISPLACEMENT RELATIONS

The displacement vector $\hat{v}$ may be expressed in reference to the spacial base vectors as

$$\hat{v} = v^i \hat{g}_i = v_i \hat{g}^i. \tag{3.1}$$

The Cauchy–Green strain tensor in terms of the displacement components is defined by

$$\gamma_{ij} = \tfrac{1}{2}(v_{i|j} + v_{j|i} + v^k{}_{|i} v_{k|j}), \tag{3.2}$$

where a single vertical line denotes covariant differentiation with respect to θ^i using the spacial metric tensors g_{ij}, g^{ij}. In the alternative forms, the displacement vector may be referred to the surface base vectors as

$$\begin{aligned}\hat{v} &= \bar{v}^\alpha(\theta^1, \theta^2, \theta^3)\hat{a}_\alpha + \bar{v}^3(\theta^1, \theta^2, \theta^3)\hat{a}_3 \\ &= \bar{v}_\alpha(\theta^1, \theta^2, \theta^3)\hat{a}^\alpha + \bar{v}_3(\theta^1, \theta^2, \theta^3)\hat{a}^3.\end{aligned} \tag{3.3}$$

The shifted components $\bar{v}_\alpha$ and $\bar{v}_3$ of the displacement vector can be expressed in power series expansion by

$$\bar{v}_\alpha = \sum_{n=0}^{\infty} \overset{(n)}{\bar{v}}_\alpha (\theta^3)^n, \qquad \bar{v}_3 = \sum_{n=0}^{\infty} \overset{(n)}{\bar{v}}_3 (\theta^3)^n, \tag{3.4}$$

and similarly the strain components by

$$\gamma_{\alpha\beta} = \sum_{n=0}^{\infty} \overset{(n)}{\gamma}_{\alpha\beta}(\theta^3)^n, \qquad \gamma_{\alpha 3} = \sum_{n=0}^{\infty} \overset{(n)}{\gamma}_{\alpha 3}(\theta^3)^n, \qquad \gamma_{33} = \sum_{n=0}^{\infty} \overset{(n)}{\gamma}_{33}(\theta^3)^n. \tag{3.5}$$

Introducing the following relations between space and surface tensors[1],

$$\begin{aligned}
&v_{\alpha|\beta} = \mu^\lambda_\alpha(\bar{v}_{\lambda\|\beta} - b_{\lambda\beta}\bar{v}_3), \qquad v_{\alpha|3} = \mu^\lambda_\alpha \bar{v}_{\lambda,3}, \\
&v^\alpha{}_{|\beta} = (\mu^{-1})^\alpha_\lambda(\bar{v}^\lambda{}_{\|\beta} - b^\lambda_\beta \bar{v}_3), \qquad v^\alpha{}_{|3} = (\mu^{-1})^\alpha_\lambda \bar{v}^\lambda{}_{,3}, \\
&v_{3|\alpha} = \bar{v}_{3,\alpha} + b^\lambda_\alpha \bar{v}_\lambda, \qquad v^3{}_{|\alpha} = \bar{v}_{3,\alpha} + b_{\alpha\lambda}\bar{v}^\lambda, \\
&v_{3|3} = v^3{}_{|3} = \bar{v}_{3\|3} = \bar{v}^3{}_{\|3} = \bar{v}_{3,3} = \bar{v}^3{}_{,3},
\end{aligned} \tag{3.6}$$

components of the Cauchy–Green strain tensors can be derived from equations (3.2), (3.4), (3.5) and (3.6) as follows:

$$\overset{(n)}{\gamma}_{\alpha\beta}=\frac{1}{2}\Bigg[\left(\overset{(n)}{\bar v}_{\alpha\|\beta}+\overset{(n)}{\bar v}_{\beta\|\alpha}-2b_{\alpha\beta}\overset{(n)}{\bar v}_3\right)-\left\{b^\lambda_\alpha\left(\overset{(n-1)}{\bar v}_{\lambda\|\beta}-b_{\lambda\beta}\overset{(n-1)}{\bar v}_3\right)+b^\lambda_\beta\left(\overset{(n-1)}{\bar v}_{\lambda\|\alpha}-b_{\lambda\alpha}\overset{(n-1)}{\bar v}_3\right)\right\}$$

$$+\sum_{m=0}^{n}\left\{\left(\overset{(m)}{\bar v}{}^\lambda{}_{\|\alpha}-b^\lambda_\alpha\overset{(m)}{\bar v}_3\right)\left(\overset{(n-m)}{\bar v}_{\lambda\|\beta}-b_{\lambda\beta}\overset{(n-m)}{\bar v}_3\right)+\left(\overset{(m)}{\bar v}_{3,\alpha}+b^\lambda_\alpha\overset{(m)}{\bar v}_\lambda\right)\left(\overset{(n-m)}{\bar v}_{3,\beta}+b^\nu_\beta\overset{(n-m)}{\bar v}_\nu\right)\right\}\Bigg],$$

$$\overset{(n)}{\gamma}_{\alpha 3}=\frac{1}{2}\Bigg[(n+1)\overset{(n+1)}{\bar v}_\alpha-nb^\lambda_\alpha\overset{(n)}{\bar v}_\lambda+\left(\overset{(n)}{\bar v}_{3,\alpha}+b^\lambda_\alpha\overset{(n)}{\bar v}_\lambda\right) \tag{3.7}$$

$$+\sum_{m=0}^{n+1}m\left\{\left(\overset{(n-m+1)}{\bar v}{}^\lambda{}_{\|\alpha}-b^\lambda_\alpha\overset{(n-m+1)}{\bar v}_3\right)\overset{(m)}{\bar v}_\lambda+\left(\overset{(n-m+1)}{\bar v}_{3,\alpha}+b^\lambda_\alpha\overset{(n-m+1)}{\bar v}_\lambda\right)\overset{(m)}{\bar v}_3\right\}\Bigg],$$

$$\overset{(n)}{\gamma}_{33}=\frac{1}{2}\left[2(n+1)\overset{(n+1)}{\bar v}_3+\sum_{m=0}^{n+1}m(n-m+2)\left(\overset{(n-m+2)}{\bar v}{}^\lambda\overset{(m)}{\bar v}_\lambda+\overset{(n-m+2)}{\bar v}_3\overset{(m)}{\bar v}_3\right)\right],$$

where a double vertical line denotes covariant differentiation with respect to θ^α using the surface metric tensors $a_{\alpha\beta}$, $a^{\alpha\beta}$ and a comma, partial differentiation with respect to θ^i. In the above derivation, the following relation[1] has been used:

$$(\mu^{-1})^\alpha_\beta=\sum_{n=0}^{\infty}(b^n)^\alpha_\beta(\theta^3)^n, \tag{3.8}$$

where

$$(b^0)^\alpha_\beta=\delta^\alpha_\beta,\quad (b^1)^\alpha_\beta=b^\alpha_\beta,\quad \cdots,\quad (b^n)^\alpha_\beta=b^\lambda_\beta(b^{n-1})^\alpha_\lambda=b^\alpha_\lambda(b^{n-1})^\lambda_\beta. \tag{3.9}$$

Equations (3.7) are considered to be general expressions of the nonlinear strain–displacement relations in the three-dimensional shell body corresponding to the expressions of shifted displacement components (3.4) and the strain components (3.5). The linearized expressions of equations (3.7) coincide with the results in [17] and these expressions reduce to the results in [8] by putting n, $m=0$, 1.

4. FUNDAMENTAL EQUATIONS DERIVED FROM VARIATIONAL PRINCIPLE

Introducing the Kirchhoff stress tensor s^{ij} (which is symmetric, i.e. $s^{ij}=s^{ji}$) which does work on the Cauchy–Green strain tensor γ_{ij} and the displacement component v_i, the modified Hellinger–Reissner variational principle for the three-dimensional elastic body in terms of a natural state may be expressed as[19, 20]

$$\delta J=0,$$

$$J=\int_V[-s^{ij}\gamma_{ij}+\Sigma(\gamma_{ij})+\tfrac{1}{2}s^{ij}(v_{i|j}+v_{j|i}+v^k{}_{|i}v_{k|j})-b^iv_i]\,\mathrm{d}V \tag{4.1}$$

$$-\int_{\mathbf{Ss}}s^i_*v_i\,\mathrm{d}S-\int_{\mathbf{Sv}}s^i(v_i-v^*_i)\,\mathrm{d}S-\int_{S^++S^-}s^i_*v_i\,\mathrm{d}S,$$

where $\mathrm{d}V$ denotes the volume element; $\Sigma(\gamma_{ij})$, the strain energy function; $\mathrm{d}S$, the element of area of the external bounding surface; s^i_*, the prescribed components of the stress vector on the part **Ss**, S^+ and S^-; v^*_i, the prescribed displacements on the part **Sv** of the lateral bounding surface of the body. The surface traction s^i is given by $s^i=s^{jk}(\delta^i_k+v^i{}_{|k})n_j$, where n_j denote the components of the outward unit vector normal to the external bounding surface of the body and b^i are the components of the body force.

Using the formulae $(2.7)_1$ and (2.9), and equations (3.4)–(3.6), the terms except the second in equation (4.1) may be written as follows:

$$\int_V s^{ij}\gamma_{ij}\,\mathrm{d}V = \int_S \sum_{n=0}^{\infty}\left(\overset{(n)}{N}{}^{\alpha\beta}\,\overset{(n)}{\gamma}_{\alpha\beta} + 2\overset{(n)}{Q}{}^{\alpha}\,\overset{(n)}{\gamma}_{\alpha 3} + \overset{(n)}{T}\,\overset{(n)}{\gamma}_{33}\right)\mathrm{d}S, \tag{4.2}$$

$$\begin{aligned}\int_V \tfrac{1}{2}s^{ij}(v_{i|j} + v_{j|i} + v^k{}_{|i}v_{k|j})\,\mathrm{d}V \\ = \int_S \sum_{n=0}^{\infty}\Bigg[&\tfrac{1}{2}\overset{(n)}{N}{}^{\alpha\beta}\left(\overset{(n)}{\phi}_{\alpha\beta} + \overset{(n)}{\phi}_{\beta\alpha}\right) - \tfrac{1}{2}\overset{(n+1)}{N}{}^{\alpha\beta}\left(b^\lambda_\alpha\,\overset{(n)}{\phi}_{\beta\lambda} + b^\lambda_\beta\,\overset{(n)}{\phi}_{\alpha\lambda}\right) + n\overset{(n-1)}{Q}{}^{\alpha}\,\overset{(n)}{\bar{v}}_\alpha - nb^\lambda_\alpha\overset{(n)}{Q}{}^{\alpha}\,\overset{(n)}{\bar{v}}_\lambda \\ &+ \overset{(n)}{Q}{}^{\alpha}\,\overset{(n)}{\psi}_\alpha + n\overset{(n-1)}{T}\,\overset{(n)}{\bar{v}}_3 + \sum_{m=0}^{\infty}\Bigg\{\tfrac{1}{2}\overset{(n+m)}{N}{}^{\alpha\beta}\left(\overset{(n)}{\phi}{}^{\lambda}{}_{\cdot\alpha}\,\overset{(m)}{\phi}_{\lambda\beta} + \overset{(n)}{\psi}_\alpha\,\overset{(m)}{\psi}_\beta\right) \\ &+ m\overset{(n+m-1)}{Q}{}^{\alpha}\left(\overset{(n)}{\phi}{}^{\lambda}{}_{\cdot\alpha}\,\overset{(m)}{\bar{v}}_\lambda + \overset{(n)}{\psi}_\alpha\,\overset{(m)}{\bar{v}}_3\right) + \tfrac{1}{2}nm\overset{(n+m-2)}{T}\left(\overset{(n)}{\bar{v}}{}^{\lambda}\,\overset{(m)}{\bar{v}}_\lambda + \overset{(n)}{\bar{v}}_3\,\overset{(m)}{\bar{v}}_3\right)\Bigg\}\Bigg]\,\mathrm{d}S,\end{aligned} \tag{4.3}$$

$$\int_V b^i v_i\,\mathrm{d}V = \int_S \sum_{n=0}^{\infty}\left(\overset{(n)}{B}{}^{\alpha}\,\overset{(n)}{\bar{v}}_\alpha + \overset{(n)}{B}{}^{3}\,\overset{(n)}{\bar{v}}_3\right)\mathrm{d}S, \tag{4.4}$$

$$\int_{Ss} s^i_* v_i\,\mathrm{d}S = \int_{cs} \sum_{n=0}^{\infty}\left(\overset{(n)}{S}{}^{\alpha}_*\,\overset{(n)}{\bar{v}}_\alpha + \overset{(n)}{S}{}^{3}_*\,\overset{(n)}{\bar{v}}_3\right)\mathrm{d}c, \tag{4.5}$$

$$\int_{Sv} s^i(v_i - v^*_i)\,\mathrm{d}S = \int_{cv} \sum_{n=0}^{\infty}\left\{\overset{(n)}{S}{}^{\alpha}\left(\overset{(n)}{\bar{v}}_\alpha - \overset{(n)}{\bar{v}}{}^*_\alpha\right) + \overset{(n)}{S}{}^{3}\left(\overset{(n)}{\bar{v}}_3 - \overset{(n)}{\bar{v}}{}^*_3\right)\right\}\mathrm{d}c, \tag{4.6}$$

$$\int_{S^+ + S^-} s^i_* v_i\,\mathrm{d}S = \int_S \sum_{n=0}^{\infty}\left(\overset{(n)}{p}{}^{\alpha}\,\overset{(n)}{\bar{v}}_\alpha + \overset{(n)}{p}{}^{3}\,\overset{(n)}{\bar{v}}_3\right)\mathrm{d}S, \tag{4.7}$$

where dc is the line element along the boundary curve c, and where cs denotes the pârt of c for which the stress vector is prescribed and cv, the part of c for which the displacement vector is prescribed. In these expressions, the stress resultant tensors and other quantities are defined as follows:

$$\overset{(n)}{N}{}^{\alpha\beta} = \int_{-t/2}^{t/2}\mu s^{\alpha\beta}(\theta^3)^n\,\mathrm{d}\theta^3, \qquad \overset{(n)}{Q}{}^{\alpha} = \int_{-t/2}^{t/2}\mu s^{\alpha 3}(\theta^3)^n\,\mathrm{d}\theta^3,$$

$$\overset{(n)}{T} = \int_{-t/2}^{t/2}\mu s^{33}(\theta^3)^n\,\mathrm{d}\theta^3, \tag{4.8}$$

$$\overset{(n)}{S}{}^{\alpha} = \int_{-t/2}^{t/2}\mu\mu^\alpha_\lambda s^\lambda(\theta^3)^n\,\mathrm{d}\theta^3, \qquad \overset{(n)}{S}{}^{3} = \int_{-t/2}^{t/2}\mu s^3(\theta^3)^n\,\mathrm{d}\theta^3, \tag{4.9}$$

$$\overset{(n)}{B}{}^{\alpha} = \int_{-t/2}^{t/2}\mu\mu^\alpha_\lambda b^\lambda(\theta^3)^n\,\mathrm{d}\theta^3, \qquad \overset{(n)}{B}{}^{3} = \int_{-t/2}^{t/2}\mu b^3(\theta^3)^n\,\mathrm{d}\theta^3, \tag{4.10}$$

$$\overset{(n)}{p}{}^{\alpha} = \Big[\mu\mu^\alpha_\lambda s^\lambda_*(\theta^3)^n\Big]_{-t/2}^{t/2}, \qquad \overset{(n)}{p}{}^{3} = \Big[\mu s^3_*(\theta^3)^n\Big]_{-t/2}^{t/2}, \tag{4.11}$$

$$\overset{(n)}{\phi}_{\alpha\beta} = \overset{(n)}{\bar{v}}_{\alpha\|\beta} - b_{\alpha\beta}\,\overset{(n)}{\bar{v}}_3, \qquad \overset{(n)}{\phi}{}^{\alpha}{}_{\cdot\beta} = \overset{(n)}{\bar{v}}{}^{\alpha}{}_{\|\beta} - b^\alpha_\beta\,\overset{(n)}{\bar{v}}_3, \qquad \overset{(n)}{\psi}_\alpha = \overset{(n)}{\bar{v}}_{3,\alpha} + b^\lambda_\alpha\,\overset{(n)}{\bar{v}}_\lambda, \tag{4.12}$$

where $\overset{(n)}{p}{}^{\alpha}$ and $\overset{(n)}{p}{}^{3}$ correspond to the resultant loads measured per unit area of the middle surface in a natural state.

Introducing the expressions (4.3)–(4.7) into the variational equation (4.1), the fundamental equations can be derived:

the equations of equilibrium,

$$\delta \overset{(n)}{\bar{v}}_\beta : \Big(\overset{(n)}{N}{}^{\alpha\beta} - b^\beta_\lambda \overset{(n+1)}{N}{}^{\alpha\lambda}\Big)_{\|\alpha} - n \overset{(n-1)}{Q}{}^\beta + (n-1) b^\beta_\lambda \overset{(n)}{Q}{}^\lambda + \sum_{m=0}^{\infty} \Big\{ \Big(\overset{(n+m)}{N}{}^{\alpha\lambda} \overset{(m)}{\phi}{}^\beta{}_{.\lambda}\Big)_{\|\alpha} - b^\beta_\lambda \overset{(n+m)}{N}{}^{\alpha\lambda} \overset{(m)}{\psi}_\alpha - n \overset{(n+m-1)}{Q}{}^\lambda \overset{(m)}{\phi}{}^\beta{}_{.\lambda} + m \overset{(n+m-1)}{Q}{}^\lambda{}_{\|\lambda} \overset{(m)}{\bar{v}}{}^\beta + m \overset{(n+m-1)}{Q}{}^\lambda \overset{(m)}{\phi}{}^\beta{}_{.\lambda} - nm \overset{(n+m-2)}{T} \overset{(m)}{\bar{v}}{}^\beta \Big\} + \overset{(n)}{B}{}^\beta + \overset{(n)}{p}{}^\beta = 0,$$

$$\delta \overset{(n)}{\bar{v}}_3 : b_{\alpha\beta}\Big(\overset{(n)}{N}{}^{\alpha\beta} - b^\beta_\lambda \overset{(n+1)}{N}{}^{\alpha\lambda}\Big) + \overset{(n)}{Q}{}^\alpha{}_{\|\alpha} - n \overset{(n-1)}{T} + \sum_{m=0}^{\infty} \Big\{ b^\lambda_\alpha \overset{(n+m)}{N}{}^{\alpha\beta} \overset{(m)}{\phi}_{\lambda\beta} + \Big(\overset{(n+m)}{N}{}^{\alpha\beta} \overset{(m)}{\psi}_\beta\Big)_{\|\alpha} - n \overset{(n+m-1)}{Q}{}^\alpha \overset{(m)}{\psi}_\alpha + m \overset{(n+m-1)}{Q}{}^\alpha{}_{\|\alpha} \overset{(m)}{\bar{v}}_3 + m \overset{(n+m-1)}{Q}{}^\alpha \overset{(m)}{\psi}_\alpha - nm \overset{(n+m-2)}{T} \overset{(m)}{\bar{v}}_3 \Big\} + \overset{(n)}{B}{}^3 + \overset{(n)}{p}{}^3 = 0; \tag{4.13}$$

the boundary conditions along the boundary curve,

$$\overset{(n)}{\bar{v}}_\alpha \quad \text{or} \quad \nu_\beta \Big[\Big(\overset{(n)}{N}{}^{\alpha\beta} - b^\beta_\lambda \overset{(n+1)}{N}{}^{\alpha\lambda}\Big) + \sum_{m=0}^{\infty} \Big(\overset{(n+m)}{N}{}^{\lambda\beta} \overset{(m)}{\phi}{}^\alpha{}_{.\lambda} + m \overset{(n+m-1)}{Q}{}^\beta \overset{(m)}{\bar{v}}{}^\alpha \Big) \Big],$$

$$\overset{(n)}{\bar{v}}_3 \quad \text{or} \quad \nu_\beta \Big[\overset{(n)}{Q}{}^\beta + \sum_{m=0}^{\infty} \Big(\overset{(n+m)}{N}{}^{\alpha\beta} \overset{(m)}{\psi}_\alpha + m \overset{(n+m-1)}{Q}{}^\beta \overset{(m)}{\bar{v}}_3 \Big) \Big]; \tag{4.14}$$

the constitutive equations in terms of the stress components,

$$s^{\alpha\beta} = \frac{\partial \Sigma}{\partial \gamma_{\alpha\beta}}, \qquad s^{\alpha 3} = \frac{\partial \Sigma}{\partial \gamma_{\alpha 3}}, \qquad s^{33} = \frac{\partial \Sigma}{\partial \gamma_{33}}; \tag{4.15}$$

and the strain–displacement relations which coincide with the former result (3.7). The constitutive equations between the Kirchhoff stress resultants and the Cauchy–Green strain tensors here were left in the unexpanded forms and discussed in detail in the following section.

The load terms which consist of the boundary terms on the upper and lower surfaces are also obtained from (4.11) as follows:

$$\overset{(n)}{p}{}^\alpha = \Big[\mu\mu^\alpha_\lambda s^{\lambda 3}_*(\theta^3)^n + \mu s^{\lambda 3}_* \sum_{m=0}^{\infty} \Big(\overset{(m)}{\bar{v}}{}^\alpha{}_{\|\lambda} - b^\alpha_\lambda \overset{(m)}{\bar{v}}_3 \Big)(\theta^3)^{n+m} + \mu s^{33}_* \sum_{m=0}^{\infty} m \overset{(m)}{\bar{v}}{}^\alpha (\theta^3)^{n+m-1} \Big]_{-t/2}^{t/2},$$

$$\overset{(n)}{p}{}^3 = \Big[\mu s^{33}_*(\theta^3)^n + \mu s^{33}_* \sum_{m=0}^{\infty} m \overset{(m)}{\bar{v}}_3 (\theta^3)^{n+m-1} + \mu s^{\alpha 3}_* \sum_{m=0}^{\infty} \Big(\overset{(m)}{\bar{v}}_{3,\alpha} + b^\lambda_\alpha \overset{(m)}{\bar{v}}_\lambda \Big)(\theta^3)^{n+m} \Big]_{-t/2}^{t/2} \tag{4.16}$$

5. CONSTITUTIVE EQUATIONS

According to the natural state theory, if the response of a body is perfectly elastic and isotropic, there exists a strain energy Σ of the form[12–15]

$$\Sigma = \Sigma(\theta^i, I_1, I_2, I_3), \tag{5.1}$$

where coordinates θ^i $(i = 1, 2, 3)$ are referred to a certain natural state and I_i $(i = 1, 2, 3)$ are the strain invariants defined by

$$I_1 = \gamma^i_i, \qquad I_2 = \tfrac{1}{2}\gamma^i_j \gamma^j_i, \qquad I_3 = \tfrac{1}{3}\gamma^i_j \gamma^j_k \gamma^k_i. \tag{5.2}$$

The stress–strain relation between the Kirchhoff stress tensor s^{ij} and the Cauchy–Green strain tensor γ_{ij} are obtained in the form

$$s^{ij} = \frac{\partial \Sigma}{\partial \gamma_{ij}}, \tag{5.3}$$

which may be attributed to Cosserats. This expression may be rewritten, considering that the stress and strain tensors are symmetric, as follows:

$$s^i_j = \frac{\partial \Sigma}{\partial \gamma^j_i} = C_1\, \delta^i_j + C_2\, \gamma^i_j + C_3\, \gamma^i_k \gamma^k_j, \tag{5.4}$$

where

$$C_1 = \frac{\partial \Sigma}{\partial I_1}, \qquad C_2 = \frac{\partial \Sigma}{\partial I_2}, \qquad C_3 = \frac{\partial \Sigma}{\partial I_3}. \tag{5.5}$$

By assuming Σ to be an analytic function of the strain measures, Σ may be expressed as a power series in three strain invariants I_1, I_2 and I_3. Since the existence of a strain energy function Σ has been assumed, it follows that

$$\frac{\partial^2 \Sigma}{\partial I_i\, \partial I_j} = \frac{\partial^2 \Sigma}{\partial I_j\, \partial I_i} \qquad \text{or} \qquad \frac{\partial C_i}{\partial I_j} = \frac{\partial C_j}{\partial I_i}. \tag{5.6}$$

Coefficients C_1, C_2 and C_3 may be expressed as power series in strain invariants. Then C_3 can be expressed in the form[16]

$$C_3 = \frac{\partial \Sigma}{\partial I_3} = C_{\Delta\Gamma\Lambda} I_1{}^{\Delta} I_2{}^{\Gamma} I_3{}^{\Lambda}, \tag{5.7}$$

where Δ, Γ, $\Lambda = 0, 1, 2, \ldots$ and summation is intended even in those terms in which an index appears more than twice. With C_3 given by (5.7) and using the relations (5.6)

$$\frac{\partial C_3}{\partial I_2} = \frac{\partial C_2}{\partial I_3} = \Gamma C_{\Delta\Gamma\Lambda} I_1{}^{\Delta} I_2{}^{\Gamma-1} I_3{}^{\Lambda},$$

so that

$$C_2 = \frac{\Gamma}{\Lambda+1} C_{\Delta\Gamma\Lambda} I_1{}^{\Delta} I_2{}^{\Gamma-1} I_3{}^{\Lambda+1} + D_{\Delta\Gamma} I_1{}^{\Delta} I_2{}^{\Gamma}, \tag{5.8}$$

where the second term on the right-hand side represents an arbitrary function of integration. Similarly, with the result (5.8) and using the relations (5.6),

$$\frac{\partial C_2}{\partial I_1} = \frac{\partial C_1}{\partial I_2} = \frac{\Delta\Gamma}{\Lambda+1} C_{\Delta\Gamma\Lambda} I_1{}^{\Delta-1} I_2{}^{\Gamma-1} I_3{}^{\Lambda+1} + \Delta D_{\Delta\Gamma} I_1{}^{\Delta-1} I_2{}^{\Gamma},$$

so that

$$C_1 = \frac{\Delta}{\Lambda+1} C_{\Delta\Gamma\Lambda} I_1{}^{\Delta-1} I_2{}^{\Gamma} I_3{}^{\Lambda+1} + \frac{\Delta}{\Gamma+1} D_{\Delta\Gamma} I_1{}^{\Delta-1} I_2{}^{\Gamma+1} + E_{\Delta} I_1{}^{\Delta}, \tag{5.9}$$

where the third term on the right-hand side represents an arbitrary function of integration. Therefore, $\Delta \geqq 1$ in the coefficient E_Δ without loss of generality.

The mixed component of stress tensor (5.4) can now be expressed as follows:

$$s^i_j = \left[\frac{\Delta}{\Lambda+1} C_{\Delta\Gamma\Lambda} I_1{}^{\Delta-1} I_2{}^{\Gamma} I_3{}^{\Lambda+1} + \frac{\Delta}{\Gamma+1} D_{\Delta\Gamma} I_1{}^{\Delta-1} I_2{}^{\Gamma+1} + E_\Delta I_1{}^{\Delta}\right]\delta^i_j$$
$$+ \left[\frac{\Gamma}{\Lambda+1} C_{\Delta\Gamma\Lambda} I_1{}^{\Delta} I_2{}^{\Gamma-1} I_3{}^{\Lambda+1} + D_{\Delta\Gamma} I_1{}^{\Delta} I_2{}^{\Gamma}\right]\gamma^i_j + [C_{\Delta\Gamma\Lambda} I_1{}^{\Delta} I_2{}^{\Gamma} I_3{}^{\Lambda}]\gamma^i_k\gamma^k_j. \quad (5.10)$$

Introducing the expanded strain components (3.5) into equation (5.2), the strain invariants can be expressed as

$$I_1 = \sum_{p=0}^{\infty} \overset{(p)}{J_1}(\theta^3)^p, \qquad I_2 = \sum_{p=0}^{\infty} \overset{(p)}{J_2}(\theta^3)^p, \qquad I_3 = \sum_{p=0}^{\infty} \overset{(p)}{J_3}(\theta^3)^p, \quad (5.11)$$

where, by using the relations (2.4) and (2.5)

$$\overset{(p)}{J_1} = a^{\alpha\lambda}\sum_{q=0}^{p}(q+1)(b^q)^\beta_\lambda \overset{(p-q)}{\gamma_{\alpha\beta}} + \overset{(p)}{\gamma_{33}},$$

$$\overset{(p)}{J_2} = \tfrac{1}{2}a^{\alpha\gamma}a^{\beta\delta}\sum_{q=0}^{p}\sum_{r=0}^{p}\sum_{s=0}^{p}(q+1)(r+1)(b^q)^\lambda_\gamma(b^r)^\nu_\delta \overset{(s)}{\gamma_{\beta\lambda}} \overset{(p-q-r-s)}{\gamma_{\alpha\nu}}$$
$$+ a^{\alpha\lambda}\sum_{q=0}^{p}\sum_{r=0}^{p}(q+1)(b^q)^\beta_\lambda \overset{(r)}{\gamma_{\alpha 3}} \overset{(p-q-r)}{\gamma_{\beta 3}} + \frac{1}{2}\sum_{q=0}^{p} \overset{(q)}{\gamma_{33}} \overset{(p-q)}{\gamma_{33}}, \quad (5.12)$$

$$\overset{(p)}{J_3} = \tfrac{1}{3}a^{\alpha\sigma}a^{\beta\varepsilon}a^{\gamma\eta}\sum_{q=0}^{p}\sum_{r=0}^{p}\sum_{s=0}^{p}\sum_{t=0}^{p}\sum_{u=0}^{p}(q+1)(r+1)(s+1)(b^q)^\lambda_\sigma(b^r)^\nu_\varepsilon(b^s)^\delta_\eta \overset{(t)}{\gamma_{\alpha\nu}} \overset{(u)}{\gamma_{\beta\delta}} \overset{(p-q-r-s-t-u)}{\gamma_{\lambda\gamma}}$$
$$+ a^{\beta\varepsilon}a^{\gamma\eta}\sum_{q=0}^{p}\sum_{r=0}^{p}\sum_{s=0}^{p}\sum_{t=0}^{p}(q+1)(r+1)(b^q)^\nu_\varepsilon(b^r)^\delta_\eta \overset{(s)}{\gamma_{\beta\delta}} \overset{(t)}{\gamma_{\gamma 3}} \overset{(p-q-r-s-t)}{\gamma_{\nu 3}}$$
$$+ a^{\alpha\lambda}\sum_{q=0}^{p}\sum_{r=0}^{p}\sum_{s=0}^{p}(q+1)(b^q)^\beta_\lambda \overset{(r)}{\gamma_{\alpha 3}} \overset{(s)}{\gamma_{\beta 3}} \overset{(p-q-r-s)}{\gamma_{33}} + \frac{1}{3}\sum_{q=0}^{p}\sum_{r=0}^{p} \overset{(q)}{\gamma_{33}} \overset{(r)}{\gamma_{33}} \overset{(p-q-r)}{\gamma_{33}},$$

where the following relation has been used:

$$g^{\alpha\beta} = a^{\alpha\lambda}\sum_{p=0}^{\infty}(p+1)(b^p)^\beta_\lambda(\theta^3)^p.$$

The powers of the strain invariants can be expressed as

$$I_1{}^{\Delta} = \sum_{p=0}^{\infty} \underset{\Delta}{\overset{(p)}{K_1}}(\theta^3)^p, \qquad I_2{}^{\Gamma} = \sum_{p=0}^{\infty} \underset{\Gamma}{\overset{(p)}{K_2}}(\theta^3)^p, \qquad I_3{}^{\Lambda} = \sum_{p=0}^{\infty} \underset{\Lambda}{\overset{(p)}{K_3}}(\theta^3)^p, \quad (5.13)$$

where

$$\underset{\nabla}{\overset{(p)}{K_i}} = \overbrace{\sum_{q=0}^{p}\sum_{r=0}^{q}\cdots\sum_{s=0}^{t}}^{\nabla-1} \overbrace{\overset{(s)}{J_i}\overset{(t-s)}{J_i}\cdots\overset{(q-r)}{J_i}\overset{(p-q)}{J_i}}^{\nabla} = \overbrace{\sum_{q=0}^{p}\sum_{r=0}^{p}\cdots\sum_{s=0}^{p}}^{\nabla-1} \overbrace{\overset{(s)}{J_i}\cdots\overset{(r)}{J_i}\overset{(q)}{J_i}\overset{(p-q-r-\cdots-s)}{J_i}}^{\nabla}$$

$(i = 1, 2, 3)$ (5.14)

This formula indicates that there are ∇-factors and $(\nabla - 1)$-summations. From (5.14) the following relation in the recurrence form may be obtained:

$$\underset{\nabla}{\overset{(p)}{K_i}} = \sum_{q=0}^{p} \underset{\nabla-1}{\overset{(q)}{K_i}} \overset{(p-q)}{J_i}. \qquad (i = 1, 2, 3) \quad (5.15)$$

The product of the powers of the strain invariants can be expressed as

$$I_1{}^{\Delta}I_2{}^{\Gamma} = \sum_{p=0}^{\infty} \overset{(p)}{\underset{\Delta,\Gamma}{\Omega}} (\theta^3)^p, \qquad I_1{}^{\Delta}I_2{}^{\Gamma}I_3{}^{\Lambda} = \sum_{p=0}^{\infty} \overset{(p)}{\underset{\Delta,\Gamma,\Lambda}{\Psi}} (\theta^3)^p, \tag{5.16}$$

where

$$\overset{(p)}{\underset{\Delta,\Gamma}{\Omega}} = \sum_{q=0}^{p} \overset{(q)}{\underset{\Delta}{K_1}} \overset{(p-q)}{\underset{\Gamma}{K_2}}, \qquad \overset{(p)}{\underset{\Delta,\Gamma,\Lambda}{\Psi}} = \sum_{q=0}^{p}\sum_{r=0}^{p} \overset{(q)}{\underset{\Delta}{K_1}} \overset{(r)}{\underset{\Gamma}{K_2}} \overset{(p-q-r)}{\underset{\Lambda}{K_3}}. \tag{5.17}$$

Using (5.13), (5.16) and (3.5), the constitutive equations in terms of the stress components can be obtained from (5.10):

$$s^{\alpha\beta} = \sum_{p=0}^{\infty} \overset{(p)}{\Xi}{}^{\alpha\beta}(\theta^3)^p, \qquad s^{\alpha 3} = \sum_{p=0}^{\infty} \overset{(p)}{\Xi}{}^{\alpha 3}(\theta^3)^p, \qquad s^{33} = \sum_{p=0}^{\infty} \overset{(p)}{\Xi}{}^{33}(\theta^3)^p, \tag{5.18}$$

where

$$\begin{aligned}
\overset{(p)}{\Xi}{}^{\alpha\beta} &= a^{\lambda\beta} \sum_{q=0}^{p} \overset{(p-q)}{\Pi_1} (q+1)(b^q)^{\alpha}_{\lambda} + a^{\sigma\lambda}a^{\varepsilon\nu} \sum_{q=0}^{p}\sum_{r=0}^{p}\sum_{s=0}^{p} \overset{(q)}{\Pi_2}(r+1)(s+1)(b^r)^{\alpha}_{\sigma}(b^s)^{\beta}_{\varepsilon} \overset{(p-q-r-s)}{\gamma_{\lambda\nu}} \\
&+ a^{\sigma\lambda}a^{\varepsilon\nu}a^{\eta\delta} \sum_{q=0}^{p}\sum_{r=0}^{p}\sum_{s=0}^{p}\sum_{t=0}^{p}\sum_{u=0}^{p} \overset{(q)}{\Pi_3}(r+1)(s+1)(t+1)(b^r)^{\alpha}_{\sigma}(b^s)^{\beta}_{\varepsilon}(b^t)^{\gamma}_{\eta} \overset{(u)}{\gamma_{\gamma\lambda}} \overset{(p-q-r-s-t-u)}{\gamma_{\delta\nu}} \\
&+ a^{\sigma\lambda}a^{\varepsilon\nu} \sum_{q=0}^{p}\sum_{r=0}^{p}\sum_{s=0}^{p}\sum_{t=0}^{p} \overset{(q)}{\Pi_3}(r+1)(s+1)(b^r)^{\alpha}_{\sigma}(b^s)^{\beta}_{\varepsilon} \overset{(t)}{\gamma_{\lambda 3}} \overset{(p-q-r-s-t)}{\gamma_{\nu 3}}, \\
\overset{(p)}{\Xi}{}^{\alpha 3} &= a^{\lambda\beta} \sum_{q=0}^{p}\sum_{r=0}^{p} \overset{(q)}{\Pi_2}(r+1)(b^r)^{\alpha}_{\lambda} \overset{(p-q-r)}{\gamma_{\beta 3}} \\
&+ a^{\lambda\beta}a^{\nu\delta} \sum_{q=0}^{p}\sum_{r=0}^{p}\sum_{s=0}^{p}\sum_{t=0}^{p} \overset{(q)}{\Pi_3}(r+1)(s+1)(b^r)^{\alpha}_{\lambda}(b^s)^{\gamma}_{\nu} \overset{(t)}{\gamma_{\gamma\beta}} \overset{(p-q-r-s-t)}{\gamma_{\delta 3}} \\
&+ a^{\lambda\beta} \sum_{q=0}^{p}\sum_{r=0}^{p}\sum_{s=0}^{p} \overset{(q)}{\Pi_3}(r+1)(b^r)^{\alpha}_{\lambda} \overset{(s)}{\gamma_{\beta 3}} \overset{(p-q-r-s)}{\gamma_{33}}, \\
\overset{(p)}{\Xi}{}^{33} &= \overset{(p)}{\Pi_1} + \sum_{q=0}^{p} \overset{(q)}{\Pi_2} \overset{(p-q)}{\gamma_{33}} + a^{\lambda\beta} \sum_{q=0}^{p}\sum_{r=0}^{p}\sum_{s=0}^{p} \overset{(q)}{\Pi_3}(r+1)(b^r)^{\alpha}_{\lambda} \overset{(s)}{\gamma_{\alpha 3}} \overset{(p-q-r-s)}{\gamma_{\beta 3}} \\
&+ \sum_{q=0}^{p}\sum_{r=0}^{p} \overset{(q)}{\Pi_3} \overset{(r)}{\gamma_{33}} \overset{(p-q-r)}{\gamma_{33}}.
\end{aligned} \tag{5.19}$$

In (5.19) the following coefficients are used:

$$\begin{aligned}
\overset{(p)}{\Pi_1} &= \frac{\Delta}{\Lambda+1} C_{\Delta\Gamma\Lambda} \overset{(p)}{\underset{\Delta-1,\Gamma,\Lambda+1}{\Psi}} + \frac{\Delta}{\Gamma+1} D_{\Delta\Gamma} \overset{(p)}{\underset{\Delta-1,\Gamma+1}{\Omega}} + E_{\Delta} \overset{(p)}{\underset{\Delta}{K_1}}, \\
\overset{(p)}{\Pi_2} &= \frac{\Gamma}{\Lambda+1} C_{\Delta\Gamma\Lambda} \overset{(p)}{\underset{\Delta,\Gamma-1,\Lambda+1}{\Psi}} + D_{\Delta\Gamma} \overset{(p)}{\underset{\Delta,\Gamma}{\Omega}}, \\
\overset{(p)}{\Pi_3} &= C_{\Delta\Gamma\Lambda} \overset{(p)}{\underset{\Delta,\Gamma,\Lambda}{\Psi}}.
\end{aligned} \tag{5.20}$$

The constitutive equations in terms of the stress resultants can be obtained by substituting (5.18) into (4.8) as follows:

$$
\begin{aligned}
\overset{(n)}{N}{}^{\alpha\beta} &= \sum_{p=0}^{\infty} \overset{(p)}{\Xi}{}^{\alpha\beta}\,[\mathrm{f}(n+p+1) - 2H\mathrm{f}(n+p+2) + K\mathrm{f}(n+p+3)],\\
\overset{(n)}{Q}{}^{\alpha} &= \sum_{p=0}^{\infty} \overset{(p)}{\Xi}{}^{\alpha 3}\,[\mathrm{f}(n+p+1) - 2H\mathrm{f}(n+p+2) + K\mathrm{f}(n+p+3)],\\
\overset{(n)}{T} &= \sum_{p=0}^{\infty} \overset{(p)}{\Xi}{}^{33}\,[\mathrm{f}(n+p+1) - 2H\mathrm{f}(n+p+2) + K\mathrm{f}(n+p+3)],
\end{aligned}
\tag{5.21}
$$

where the following notation is used:

$$
\Big[(\theta^3)^n\Big]_{-t/2}^{t/2} = \left(\frac{t}{2}\right)^n [1-(-1)^n] \equiv n\mathrm{f}(n),
\tag{5.22}
$$

$$
n\mathrm{f}(n) = \begin{cases} 0, & \text{for even } n,\\ t^n/2^{n-1}, & \text{for odd } n. \end{cases}
$$

The expressions (5.21) are considered to be the general constitutive equations in terms of stress resultants and strain components for elastic shells of isotropic materials. These expressions involve two types of power series in the strain components $\overset{(n)}{\gamma}_{ij}$ and the mixed components of the fundamental tensor of the middle surface b^{α}_{β}. In order to examine the constitutive equations and to introduce various approximations, it is convenient to show the powers in these expressions. The notation $1/R$ (R is the least principal radius of curvature) will be used to represent terms of the type b^{α}_{β} and γ to represent terms of the type $\overset{(n)}{\gamma}_{ij}$. The results can be summarized as Table 1.

Table 1

	$\overset{(p)}{J}_1$	$\overset{(p)}{J}_2$	$\overset{(p)}{J}_3$	$\underset{\Delta}{\overset{(p)}{K}_1}$	$\underset{\Gamma}{\overset{(p)}{K}_2}$	$\underset{\Lambda}{\overset{(p)}{K}_3}$	$\underset{\Delta,\Gamma}{\overset{(p)}{\Omega}}$	$\underset{\Delta,\Gamma,\Lambda}{\overset{(p)}{\Psi}}$
γ	1	2	3	Δ	2Γ	3Λ	$\Delta+2\Gamma$	$\Delta+2\Gamma+3\Lambda$
$\frac{1}{R}$	p	p	p	p	p	p	p	p

Since Table 1 shows that the powers of γ are independent of those of $1/R$, only the powers of γ are considered in the following. In the expressions (5.20), the power to which γ occurs in terms involving the material coefficients $C_{\Delta\Gamma\Lambda}$, $D_{\Delta\Gamma}$ and E_{Δ}, which are generally considered to be functions of θ^i, is determined by applying the result of Table 1 to (5.20) and the result is shown in Table 2.

Table 2

	$\overset{(p)}{\Pi}_1$			$\overset{(p)}{\Pi}_2$		$\overset{(p)}{\Pi}_3$
	$C_{\Delta\Gamma\Lambda}$	$D_{\Delta\Gamma}$	E_{Δ}	$C_{\Delta\Gamma\Lambda}$	$D_{\Delta\Gamma}$	$C_{\Delta\Gamma\Lambda}$
γ	$\Delta+2\Gamma+3\Lambda+2$	$\Delta+2\Gamma+1$	Δ	$\Delta+2\Gamma+3\Lambda+1$	$\Delta+2\Gamma$	$\Delta+2\Gamma+3\Lambda$

Finally, the power to which γ occurs in terms involving the material coefficients in the constitutive equations (5.21) can be found from the result of Table 2 and equations (5.21):

$$\begin{aligned} C_{\Delta\Gamma\Lambda} &: \Delta + 2\Gamma + 3\Lambda + 2, \\ D_{\Delta\Gamma} &: \Delta + 2\Gamma + 1, \\ E_{\Delta} &: \Delta. \end{aligned} \tag{5.23}$$

In the case of approximate constitutive equations of p-degree, summation is to be executed over all the combinations of Δ, Γ and Λ that are obtained by equating each power of (5.23) to p. Denoting the stress resultants of p-degree by $\overset{(n)}{N}{}^{\alpha\beta}_{(p)}$, $\overset{(n)}{Q}{}^{\alpha}_{(p)}$ and $\overset{(n)}{T}_{(p)}$, the approximate constitutive equations of the kth order may be derived from

$$\underset{(k)}{\overset{(n)}{N}}{}^{\alpha\beta} = \sum_{p=1}^{k} \overset{(n)}{N}{}^{\alpha\beta}_{(p)}, \qquad \underset{(k)}{\overset{(n)}{Q}}{}^{\alpha} = \sum_{p=1}^{k} \overset{(n)}{Q}{}^{\alpha}_{(p)}, \qquad \underset{(k)}{\overset{(n)}{T}} = \sum_{p=1}^{k} \overset{(n)}{T}_{(p)}. \tag{5.24}$$

Explicit expressions of the stress resultants of p-degree in terms of the expanded strain components are quite complicated and will not be shown here. The power of $1/R$ in (5.21) is $(p + 2)$ because of the presence of μ which has a second order term in $1/R$.

6. CONCLUSION

A rigorous derivation of the complete and consistent two-dimensional shell theory incorporating the geometrical and physical nonlinearities has been obtained from the three-dimensional theory of elasticity.

Based on the expressions of the displacement components (3.4), a fully consistent set of fundamental shell equations for geometrically nonlinear problems has been derived systematically through the modified Hellinger–Reissner variational principle without introducing the Love–Kirchhoff hypothesis. All these expressions are referred to a certain natural state and given in terms of stress resultants defined by (4.8) and the expanded Cauchy–Green strain tensors. For elastic shells of isotropic materials, assuming the existence of a strain energy function Σ given by a power series in three strain invariants, the general constitutive equations in the Cosserats form have been derived as power series which involve two types of power series in the strain components $\overset{(n)}{\gamma}_{ij}$ and the mixed components of the second fundamental tensor of the middle surface b^{α}_{β}. Thus, the theory presented here can describe the large elastic deformations of shells undergoing finite strains. The classical shell theories are directly derivable from the present results by proper truncations of the series.

Although the fundamental equations derived here are very complex in their forms for practical uses, the following assumptions, i.e.

(1) Love–Kirchhoff hypothesis,
(2) small strains and/or small displacements,
(3) small curvatures and/or small rotations, etc.

which have always been introduced in the classical shell theories, are not included. Therefore, on the basis of the present results and introducing various assumptions, it will be possible to estimate the accuracy of the earlier shell theories.

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Абстракт — В работе дается общая нелинейная теория оболочек для больших прогибов и конечных деформаций, относительно некоторого естественного состояния. Путем разложения компонентов перемещений в степенные ряды по координате θ^3, нормальной к недеформированной срединной поверхности оболочек, разложения в ряды для тензоров деформации Коши-Грина выражаются с помощью этих разложенных компонентов перемещений. Посредством преобразованного вариационного принципа Геллингера-Райснера для трёхмерной упругой сплошной среды, определяется система основных уравнений оболочки, в виде разложенных в рады тензоров деформации Коши-Грина и сумм напряжений Кирхгоффа. Не предполагается гипотезу Лява-Кирхгоффа. Принимаются во внимание растяжение высшего порядка и изгиб. Для упругих оболочек из изотропного материала, выводятся общие нелинейные уравнения состояния, предполагая, что энергия деформации является аналитической функцией меры деформации. Таким образом, создавается полная и совместна двухмерная теория оболочек, заключающая геометрические и физические нелинейности. Классические теории можно непосредственно получить из предложенных результатов, путем правильного отбрасывания членов рядов.